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Physical conditions in the analytical solution on the Mathieu equation

Condiciones físicas en la solución analítica de la ecuación de Mathieu

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Abstract

The study aims to analyze the Mathieu equation associated with linear systems under properties, use the one-dimensional Floquet theorem and explain the behavior of the solutions of the Mathieu equation and function relating to the Hill equation together with the theory of disturbances. The most common mathematical problem in physics is finding solutions to certain second-order differential equations subject to boundary conditions. A hypothetical deductive approach method, handling qualitative equations of mathematical physics, was adopted in review. With Mathieu's equation, the existence of the physical conditions that influence the value of the parameters that define the equation were answered. It is important due to its usefulness in the field of analytical and mechanical dynamics, physical systems subjected to parametric excitation; In addition, it leaves other variations open for new applications.

Keywords: boundary problems, mathieu equation, hill equation, perturbation theory

Resumen

El estudio tiene como objetivo analizar la ecuación de Mathieu asociado a sistemas lineales bajo propiedades, utilizar el teorema Floquet de una dimensión y explicar el comportamiento de las soluciones de la ecuación y función de Mathieu relacionado con la ecuación de Hill junto a la teoría de perturbaciones. Los problemas matemáticos más comunes en física son encontrar soluciones de ciertas ecuaciones diferenciales de segundo orden sujetas a condiciones de frontera. Se adoptó en revisión, un método enfoque hipotético deductivo, manejo de ecuaciones cualitativas de la física matemática. Con la ecuación de Mathieu, se dio respuesta a la existencia de las condiciones físicas que influyen para obtener el valor de los parámetros que definen la ecuación. Tiene importancia por su utilidad en el ámbito de la dinámica analítica y mecánica, sistema físico sometidas a excitación paramétrica; además, deja abierta otras variaciones para nuevas aplicaciones.

Palabras clave: problemas de contorno, ecuación de mathieu, ecuación de hill, teoría de perturbaciones

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INTRODUCTION

Most of the functions of physics-mathematics have their origin in research problems. Mathieu functions were introduced when determining the modes of vibration of a stretching membrane having an elliptical boundary. The two-dimensional wave equation was transformed into $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + k^2 V = 0$ confocal elliptic coordinates, and separated into two ordinary differential equations of the form:

$$\frac{d^2x}{dt^2} + \big(b - 2qcos(2t)\big)x = 0 \qquad (1)$$

$$\frac{d^2x}{dt^2} - \big(b - 2qcos(2t)\big)x = 0 \qquad (2)$$

being b, q real parameters.

Equation (2) was solved with substitution of $\pm it$ by t and vice versa. For $q > 0$ (1) is the Mathieu equation and (2) is the modified Mathieu equation; However, in the elliptical membrane problem, the appropriate solutions to (2) are called Mathieu , periodic in t with period π or 2π, as a consequence of this periodicity, b has special values called characteristic numbers. The solutions of equation (2) that correspond to the same solutions of (1) for the value of , are called modified b Mathieu functions that are obtained from the functions using imaginary arguments.

The first solutions of integral order of equation (1) were given through series of cosines and sines, which satisfy the conditions to be Fourier series, although the coefficients are not obtained by ordinary integration (Hille, 1923).

Mathieu's equation, that is $\frac{d^2x}{dt^2} + [b - 2q\varphi(2t)]x = 0$, where $-2q\varphi(2t) = 2[\theta_2 cos2t + \theta_4 cos3t + \cdots], b = \theta_0$, being θ a known parameter.

In 1883, Floquet published a general work on linear differential equations with periodic coefficients, of which Hill's and Mathieu 's equations are particular cases. The first appearance of an asymptotic formula for modified Mathieu functions was given by Maclaurin; However, none of these authors obtained the multiplicative constants, which are necessary for numerical work (Hartman, 1964; Shingareva, 1995).

Marshall (1909) obtained the multipliers for the series, while Hilbert studied the characteristic values and obtained an integral equation with discontinuous nuclei for the periodic solutions of equation (1). In this same line, Sieger published an article on the diffraction of electromagnetic waves by an elliptical cylinder and worked with orthogonality, using an integral equation with a different nucleus, and obtained a solution of equation (2) as a series of products of Bessel functions, and the analysis of their convergence.

A systematic study was given by Whittaker (1914), and he found an integral equation for a set of periodic functions of integral order; Using this method as a basis, Young (1914) provided a method for finding general solutions and discussed the problem of stability; That is, if the solution tends to zero or infinity when $t \to \infty$, the recurrence formulas for the Mathieu functions were arrived at (Arnold, 1973).

A general study of the Mathieu equation is due to J. Dougall who obtained asymptotic expansions for the Mathieu functions modified with $t$large, a boundary integral, which under certain conditions, degenerated to an integral of the Bessel function. Until 1921, the only known periodic solutions of the Mathieu equation (1) had period π y 2π. Poole recently generalized the situation and showed that with appropriate values of b for an q assigned, equation (1) can admit solutions that have period $2n\pi$, where n is an integer greater than or equal to two (Zimmerman, 1995).

The second solution of equation (1), being a characteristic number for a periodic solution of integral order, was recently studied covering various aspects of convergence, and integral equations for the second solution, using expansions in the b ordinary and modified Mathieu functions, reproduced Rayleigh 's formula for diffraction of electromagnetic waves (Hille, 1923).

In the case of an elliptical membrane, the solution is expressed in terms of the product of ordinary and modified Mathieu functions. The zeros of the modified Mathieu functions determine the vibrational angular frequencies and confocal nodal ellipses, while the modified Mathieu functions define a system of confocal nodal hyperbolas. When the eccentricity of the bounded ellipse tends to zero, the nodal ellipse tends to nodal circles, and the nodal hyperbola tends to nodal radii (Cesari, 1963).

THEORETICAL FRAMEWORK

For the work we have linear differential equations of the form $\frac{dx}{dt} = B(t)x$, where $B(t)$ is a real, non-singular square matrix and $x = (x_1, x_2, \dots, x_n)$ an n-dimensional vector. Linear differential equations that appear naturally in the study of nonlinear equations, $\dot{x} = X(x,t)$ where $X(x,t) = (X_1(x,t), X_2(x,t), \dots, X_n(x,t))$ is an n-dimensional vector function of x y t. If X it is not a function of time, then an expansion around fixed points, defined by the equation $X(x) = 0$, normally gives equations with constant matrix B, and the study of these linear equations provides important information about the stability of those fixed points (Ross, 1995).

Alternatively, if the nonlinear system admits a periodic solution $\delta(t)$, $\delta(t) = \delta(t+T)$, for any t and all T greater than zero, then an expansion of $\delta(t)$, leads to an equation with the matrix $B(t)$, T-periodic in time.

Solution in terms of a fundamental matrix: the solution with the initial condition $x(t_0) = x_0$ can be expressed as $x(t) = \Phi(t)\Phi(t_0)^{-1}x(t_0)$ where $\Phi(t)$ is any fundamental matrix. The matrix $U(t_2,t_1) = \Phi(t_2)\Phi(t_1)^{-1}$, which depends on two times t_1 and t_2 is called the propagation matrix , this matrix takes the solution from one time t_1 to t_2.

Floquet–Lyapunov theorem: for the equation of $\dot{x} = B(t)x$, where the elements of the matrix $B(t)$, are functions of t, T —periodic and piecewise continuous, with a number of discontinuities in $(-\infty, +\infty)$ and integrable in each discontinuity, a fundamental matrix is expressed in the form $\Phi(t) = P(t)e^{kt}$ where $P(t)$ is a square, non-singular matrix, T periodic for everything t and with continuous elements, with continuous derivatives by parts and integrables, (Barrow, 1996).

Perturbation theory: by applying perturbation theory, approximate analytical solutions of equations of various types are obtained, which involve a small parameter. Perturbation theory is a collection of methods for obtaining approximate analytical solutions of equations involving a small parameter ε, taking into account that most methods of modern physics contain applications of perturbation theory.

It is very common in applications, when a model of a physical, chemical, or biological system is studied, we have the following equation, which involves a small parameter ε defined in an interval $I = (0; c_0)$: $F(x,\varepsilon) = 0$ where $x$is the real variable, it is called a perturbed solution. If we know the unperturbed solution, that is, the equation $F(x,0) = 0$, the asymptotic analysis leads to constructing the approximate analytical solution for $\varepsilon > 0$ small (Bickley, 1940).

Fundamental theorem of perturbation theory: if $B_0 f_0(\delta) + B_1 f_1(\delta) + B_2 f_2(\delta) + \cdots + B_n f_n(\delta) + 0\left(f_{n+1}(\delta)\right) = 0$ where $f_n(\delta)$ is an asymptotic sequence and the coefficients B_i $(i = 0,1,2,3, \ldots, n)$ are independent of δ, then $B_0 = B_1 = B_2 = \cdots = B_n = 0$.

Floquet theory: for systems with periodic coefficients, $\frac{dx}{dt} = A(t)x,\ A(t+T) = A(t)$ for all t, where $A(t)$ is a real, non-singular matrix, with elements that are T-periodic functions of t. In general, solutions of periodic linear systems cannot be expressed in terms of elementary functions, but linearity and periodicity mean $A(t)$ that the behavior of a solution for all times can be deduced from the general solution over a finite interval of length T. This property means that the behavior of solutions when $t \to \infty$ can often be deduced from analytical approximations or numerical solutions (Jordan, 1999).

Periodic multidimensional systems: for the homogeneous linear equation of order n, $\frac{dx}{dt} = A(t)x,\ \ A(t+T) = A(t)$ for all t, where $A(t)$ is a square matrix, whose elements are T-periodic functions, assuming that this equation has n linearly independent solutions $\{x_1(t), x_2(t), x_3(t), \ldots, x_n(t)\}$, it is possible to form the fundamental matrix $\Phi(t)$ that satisfies the equation $\frac{d\Phi}{dt} = A(t)\Phi$. The vectors of the form $y_k(t) = x_k(t+T)$ are also solutions of the original differential equation, and we have $\frac{dy_k}{dt} = \frac{dx_k(t+T)}{d(t+T)} = A(t+T)x_k(t+T) = A(t)y_k$. Therefore, $y_k(t)$ it must be a linear combination of the $x_j(t), j = 1,2, \ldots, n$, that is $y_k(t) = \sum_{j=1}^{n} x_j(t)e_{jk}$ for some constants e_{jk}.

These new solutions can be used to form the fundamental matrix $\Phi(t+T)$, and we have $\Phi(t+T) = \Phi(t)E$, where $E$is a matrix with elements e_{jk}, since $det\ det\ \left(\Phi(t+T)\right)\ = det\ det\ \left(\Phi(t)\right)\ det\ (E)$y $det\ det\ \left(\Phi(t)\right) \neq 0,\ E$ is non-singular.

Importance of eigenvalues and eigenvectors: If λ is an eigenvalue and a the associated eigenvector, $Ea = \lambda a$, then the solution $z(t) = \Phi(t)a$ has the property $z(t+T) = \lambda z(t)$, for all t, since $z(t+T) = \Phi(t)Ea = \lambda\Phi(t)a = \lambda z(t)$, for all t.

The eigenvalues of the matrix $E$are called characteristic multipliers or characteristic numbers of the system; The eigenvalues of E are independent of the choice of the fundamental matrix, therefore, they are a property of the system, not of any particular solution, if the different fundamental matrices $\Phi_1(t)$ and $\Phi_2(t)$ give rise to the matrices E_1 and E_2, since $\Phi_2(t) = \Phi_1(t)C$, for some constant C, We also $E_2 = C^{-1}E_1C$ have $\Phi_2(t+T) = \Phi_1(t)E_1C = \Phi_2(t)C^{-1}E_1C$.

Similarity: E_1 and $E_2$are similar and have the same eigenvalues, then it is convenient to write the eigenvalues in the form $\lambda_k = e^{T\rho_k},\ \rho_k \epsilon C$ where ρ_k it is made unique by choosing its imaginary part, and which satisfy $-\pi < 0(T\rho_k) \leq \pi$, therefore, $\rho_k = \frac{1}{T} ln(\lambda_k)$ or $T\rho_k = ln|\lambda_k| + iarg(\lambda_k),\ -\pi < arg\ (\lambda_k) \leq \pi$, where lnx is the main branch of the natural logarithm. The ρ_k are called the characteristic exponents. If E it has n distinct eigenvalues λ_k, the equation has n solutions linearly of the form $z_k(t) = p_k(t)e^{\rho_k t}$ where $p_k(t)$ is a periodic vector function of time, then $p_k(t+T) = z_k(t)e^{-\rho_k(t)} = p_k(t)$, for everything t (Edwards, 2000).

METHODOLOGY

T We follow physical aspects that generate a mathematical model, in this case, two problems that lead to Mathieu 's equation:

Hill's lunar theory: the equations of motion of two planets with mass m_1y $m_2$and the sun with mass m, moving under Newton's laws without considering all the other effects that influence them, with r_1 and r_2 being position vectors of the planets and r the radius vector position of the sun. The force on the

Sun, taking the gravitational constant equal to unity, allows us to arrive at the equation $Z'' + v^2 Z + \frac{kZ}{r^2} = 0$. The function $Z \equiv 0$ is solution of this equation; However, you should consider the case $Z \approx 0$and see how it behaves Z. Therefore, it is assumed that X, Y they are periodic functions of τ, and of t, the equation can be written $Z'' + \theta Z = 0$, where $\Theta = v^2 + \frac{k}{\tau^3}$ is a function (Hirsch, et al, 1974).

Problem of the pendulum with variable length: considering a pendulum of variable length A with a mass m, and a support that moves vertically with displacement $\delta(r)$, its Cartesian coordinates are $x = Asen\alpha$, $y = \delta(t) + Acos(\alpha)$, in order to find the equation of motion, we derive the equations to obtain the kinetic and potential energies, $x' = Acos\alpha$, $y' = \delta'(t) - Asen(\alpha)$, which is obtained $T = \frac{M}{2}[(x')^2 + (y')^2]$; consequently, $T = \frac{M}{2}\left[(\delta)'^2 + A^2(\alpha')^2 - 2A\delta'\alpha' sen(\alpha)\right]$, with potential energy $V = -Mg(\delta + Acos(\alpha)$, with support in the Lagrange equation, $\frac{d}{dt}\left(\frac{\partial T}{\partial \alpha}\right) - \frac{\partial T}{\partial \alpha} = -\frac{\partial V}{\partial \alpha}$, results $A\alpha'' + (g - \delta'')sen\alpha = 0$, which is similarly $x'' + (a - 2qcos(2t))x = 0$ called the Mathieu equation (Vvedensky, 1993; Jeffreys, 1924).

RESULTS AND DISCUSSIONS

Hill equation

Be the Equation of the form $\frac{d^2x}{dt^2} + [a + q(t)]x = 0$, where a is a constant and $q(t)$ a function of period T. A special case of this equation can be seen if $q(t) = 2pcos(2t)$y is the $T = \pi$ Mathieu equation. By doing so $y = \dot{x}$ it is possible to write the Hill equation in a standard matrix form

$$\frac{dr}{dt} = A(t)r,\ r = (\delta_1\ \delta_2\ ), A = (0\ 1\ \ -a - q(t)\ 0\ ) \qquad (3)$$

in (3) as $Tr(A) = 0$, and the equation $W(t) = W(t_0)exp\left(\int_{t_0}^{t} Tr(A(s)ds\right)$ shows that the $det\ det\ (\Phi)$ is constant and then $det\ det\ (E) = 1$ and the product of the eigenvalues is equal to one. The eigenvalues of $E$are given by $\lambda^2 - Tr(E)\lambda + 1 = 0$ then $2\lambda = Tr(E) \pm \sqrt{Tr(E)^2 - 4}$. Therefore, the long-term behavior of the solutions is determined mainly by the single real number $Tr(E)$.

Let us consider two independent solutions that satisfy the initial conditions $\gamma_1(0) = 1, \dot{\gamma}_1(0) = 0$, $\gamma_2(0) = 0$, $\dot{\gamma}_2(0) = 1$, therefore, $\Phi(0) = I$ in addition $Tr(E) = \gamma_1(T) + \dot{\gamma}_2(T)$, there are five different cases depending on the values of $Tr(E)$.

$Tr(E) > 2$. The eigenvalues are positive, different, not equal to $+1$and satisfy $0 < \lambda_1 < 1 < \lambda_2$.The characteristic exponents are $\pm\rho$, where $T_\rho = ln\lambda_2 > 0$and two linearly independent solutions are $\delta_1(t) = e^{-\rho t}q_1(t)$, $\delta_2(t) = e^{\rho t}q_2(t)$, where $q_k(t)$are periodic functions with period T.

$Tr(E) = 2$. The eigenvalues are identical and equal to $+1$; therefore $\rho = 0$. We see that the behavior of the solutions depends on the number of Eindependent eigenvectors:

The matrix E has two linearly independent eigenvectors: there are two solutions with period T and $\delta_1(t) = q_1(t)$, $\delta_2(t) = q_2(t)$,

where $q_k(t)$ are T-periodic functions.

The matrix E has an eigenvector linearly: the two independent solutions are $\delta_1(t) = q_1(t)$, $\delta_2(t) = tq_1(t) + q_2(t)$, where $q_k(t)$are T-periodic functions. The first solution is limited. The amplitude of the second solution grows linearly with t, so that there is an stable solution and an unstable solution.

$|Tr(E)| < 2$. The eigenvalues of E are complex and are written $\lambda = e^{\pm i\theta}$, where $0 < \theta < \pi$, with characteristic exponents $T_\rho = \pm icos^{-1}(\frac{Tr(E)}{2})$, consequently the two independent solutions are $\delta_1(t) = e^{i\rho t}q_1(t)$, $\delta_2(t) = e^{-i\rho t}q_2(t)$, where $q_k(t)$ they are T-periodic functions. In this case all solutions are bounded for all times t.

$Tr(E) = -2$. The eigenvalues are identical and equal to -1. Again the behavior of the solutions depends on the number of independent eigenvectors of E:

The matrix E has two linearly independent eigenvectors: there are two solutions with period $2T$ because $T_\rho = i\pi$, and the two independent solutions are $\delta_1(t) = q_1(t)$, $\delta_2(t) = q_2(t)$, also satisfy the periodicity conditions $\delta_1(t+T) = -\delta_1(t)$, $\delta_2(t+T) = -\delta_2(t)$.

The matrix E has only one linearly independent eigenvector: the two independent solutions are $\delta_1(t) = q_1(t)$, $\delta_2(t) = tq_1(t) + q_2(t)$

where $p_k(t)$are the 2T-periodic functions. The first solution is limited. The amplitude of the second solution grows linearly with t, therefore, there is a stable solution and an unstable solution.

$Tr(E) < -2$. The eigenvalues are real, negative, different from and not equal to -1 and satisfy $\lambda_2 < -1 < \lambda_1 < 0$. The characteristic exponents are $T_\rho = \pm ln(-\lambda_2) + i\pi$, and the two linearly independent solutions are $\delta_1(t) = e^{-\rho t}q_1(t)$, $\delta_2(t) = e^{\rho t}q_2(t)$, with $\delta_1(t)$ decreasing and $\delta_2(t)$ increasing when $t \to \infty$.

This form of stability classification allows us to conjecture that there is a strong distinction between stable and unstable solutions; which is true, but only for long times, or formally $t \to \infty$. If we look at a solution in finite time, the distinction is not so clear, because solutions are generally smooth, continuous differential functions that depend on the parameters of the system. Let's say that, if $Tr(E) = -2$, the amplitude of a solution grows linearly with t: yes $Tr(E) = -2 + \epsilon^2$ (Feynman, 1965).

The Mathieu equation

To illustrate the behavior analyzed previously in Hill's equation, let Mathieu 's equation be , $\frac{d^2x}{dt^2} + (a - 2qcos2t)x = 0$, where we set the parameter q, let $q = 2$, then analyzing how the solutions depend on parameters $a$we calculate $Tr(E)$. The Sturm-Liouville theory shows that for $0 \le t < 2\pi$ and periodic 2π solutions exist π at discrete values of a, which depends on q. These values are classified according to the convention $a_0(q) < b_1(q) < a_1(q) < \cdots, q \neq 0$, with $a_k$the even solutions and b_k the odd solutions being recognized . We also consider solutions for other values of a (Shingareva, 1995).

The points where $Tr(E) = 2$ they give the eigenvalues $a_0(q), \{b_{2k}(q), a_{2k}(q)\}$, $k = 1,2, \ldots$, corresponding to the Mathieu functions of period π, $x = ce_0(t,q)$, $\{se_{2k}(t,q), ce_{2k}(t,q)\}$. The points where $Tr(E) = -2$ they give the eigenvalues $\{b_{2k+}(q), a_{2k+1}(q)\}$, $k = 1, 2, 3, \ldots$, corresponding to the Mathieu functions of period 2π $\{se_{2k+1}(t,q), ce_{2k+1}(t,q)\}$.

The damped Mathieu equation

The effects of adding a small linear damped term to the Mathieu equation becomes $\frac{d^2x}{dt^2} + \upsilon\frac{dx}{dt} + (a - 2qcos2t)x = 0,\ \upsilon \ge 0$. Suppose it $\upsilon$is small, when $\upsilon = 0$, then it is a previous case. For q small regions of instability start from the points $a = n^2$on the axis a. The damped term, $\upsilon\frac{dx}{dt}$, reduces energy of the system, therefore, the damped term competes with the term that drives resonance; However, hopefully the presence of damping of the areas associated with unsteady motion will decrease. The damped Mathieu equation has the canonical form

$\frac{d}{dt}(x\ y\ ) = A(t)(x\ y\ )$, where $\frac{dx}{dy} = y, A(t) = (0\ \ -1\ 2qcos(2t) - a\ \ -\upsilon\ )$ (4)

we see that the determinant of the monodromy matrix E, in (4) is $det\ det\ (E) =\ e^{-\upsilon\pi}$, the period is π. Letting $Tr(E) = 2\theta$, the eigenvalues of E, the characteristic multipliers, are the solutions of $\lambda^2 - 2\theta\lambda + e^{-\upsilon\pi} = 0$, whence $\lambda_{\pm} = \theta \pm \sqrt{\theta^2 - e^{-\upsilon\pi}}$. The stability condition is $\left|\lambda_{\pm}\right| \leq 1$, and because $\lambda_+\lambda_- = e^{-\upsilon\pi} < 1$ we see that it is possible that both multipliers are real and that the solutions are stable.

Stability boundaries: if $\theta > 0$then $0 < \lambda_-\ \ < \lambda_+$, therefore, the stability boundary is given by the condition $\lambda_+ = 1$, i.e., $\theta = \frac{1}{2}(1 + e^{-\upsilon\pi})$. If $\theta < 0$then $\lambda_-\ \ < \lambda_+ < 0$, then the stability boundary is by the condition $\lambda_- = -1$, that is, $\theta = -\frac{1}{2}(1 + e^{-\upsilon\pi})$, it follows that the stability condition is $|Tr(E)| < 1 + e^{-\upsilon\pi}$.

If $Tr(E) = 1 + e^{-\upsilon\pi}$then $\lambda_+ = 1$, and $\lambda_- = e^{-\upsilon\pi}$ and the general theory shows that two independent solutions are $p_+(t)$, and $p_-(t)e^{-\upsilon t}$where $p_{\pm}(t)$ are π periodic functions of t. If $Tr(E) = -(1 + e^{-\upsilon\pi})$ then $\lambda_- = -1$and $\lambda_+ = -e^{-\upsilon\pi}$ and two independent solutions are $q_-(t)$, and $q_+(t)e^{-\upsilon t}$, where $q_{\pm}(t)$ are 2π periodic functions of t.

DISCUSSION

Approximate analytical solutions: Let us consider the construction of periodic approximate analytical solutions of the damped Mathieu equation using perturbation theory. Finding the expansions using perturbation theory for periodic solutions in the presence of damping is more complicated than when there is no damping, except if $a \approx 0$ or $a \approx 1$(Naytch, 1973).

The case: $a_0 = 1, A_0 = 1, a_1 = \sqrt{1 - \mu^2}$ The analysis necessary to obtain solutions through perturbation theory is algebraically tedious, so we describe the first terms of the expansions for particular cases. Complete calculations can be done using computer algebra methods (Shingareva, 1995). If we assume that υy $q$are small and initially sets $v = \mu q$ to derive an expansion with respect to the parameter q. The equation is

$$\frac{d^2x}{dt^2} + a(q)x = q\left(2xcos2t - \mu\frac{dx}{dt}\right), \upsilon = \mu q.$$

Now we write $x(t)$ and $a(q)$ as a power series in q,

$$x(t) = x_0(t) + x_1(t)q + x_2(t)q^2 + x_3q^3 + \cdots$$

$$a(q) = a_0 + a_1q + a_2q^2 + a_3q^3 + \cdots,$$

to obtain the infinite set of equations:

$$\frac{d^2x_0}{dt^2} + a_0x_0 = 0,$$

$$\frac{d^2x_1}{dt^2} + a_0x_1 = -a_1x_0 + 2x_0cos2t - \mu\frac{dx_0}{dt},$$

$$\frac{d^2x_2}{dt^2} + a_0x_2 = -a_2x_0 - a_1x_1 + 2x_1cos2t - \mu\frac{dx_1}{dt},$$

…

$$\frac{d^2x_n}{dt^2} + a_0x_n = -\sum_{k=1}^{n}\ a_kx_{n-k} + 2x_{n-1}cos2t - \mu\frac{dx_{n-1}}{dt}.$$

This method can be used if $a_0 = 0$ or $a_0 = 1$but fails if $a_0 \geq 2$. In this analysis of perturbation theory we consider the case $n = 1$, and therefore, we find an approximation for $q_{-}(t)$, the - periodic solution 2π. The solution to the equation is $x_0(t) = A_0 cost + B_0 sent$, where $A_0(t), B_0(t)$ are constants that must be determined. Substituting the previous equation on the right side of the equation for $x_1$we have, after some simplifications, $\frac{d^2x_1}{dt^2} + x_1 = [A_0(1-a_1) - \mu B_0]cost + \quad [B_o(1+a_1) - \mu A_0]sent + A_0 cos3t + B_0 sen3t$. This equation has periodic solutions only if the coefficients of $cost$ and $sent$ are zero: $A_0(1-a_1) - \mu B_0 = 0, B_0(1+a_1) - \mu A_0 = 0$.

In the undamped limit, which is the case $\mu = 0$, the equations have two solutions:

$a_1 = -1, A_0 = 0 \rightarrow x = B_0 sent$ takes us to $se_1(q,t)$

$a_1 = 1, B_0 = 0 \rightarrow x = A_0 cost$ takes us to $se_1(q,t)$

where A_0 and B_0 are chosen to satisfy appropriate normalization conditions.

If $\mu > 0$ homogeneous equations have a single solution only if the determinant of the coefficients is zero, this means that $a_1^2 = 1 - \mu^2$ it leads to two solutions

$a_1 = -\sqrt{1-\mu^2}, A_0 = \frac{\mu B_0}{1+\sqrt{1-\mu^2}}$,

$a_1 = \sqrt{1-\mu^2}, B_0 = \frac{\mu A_0}{1+\sqrt{1-\mu^2}}$.

But how $a_0 = 1$ the equation appears, in terms of the original variable $v = \mu q$, the expressions: $a = 1 \pm \sqrt{q^2 - v^2}$or $q^2 - (a-1)^2 = v^2$, which are the equation of a hyperbola. This analysis shows that yes $0 \leq q < v$and $a \approx 1$ all solutions are stable, so for q small, solutions that are unstable when $v = 0$ become stable when $v > 0$.

We now return to the derivation of the expansion of the problem. The only solutions are obtained by setting the coefficient of $cost$ or $sent$. The solution that reduces to $se_1(t,q)$ when $v = 0$ is obtained by leaving fixed $a_1 = -\sqrt{1-\mu^2}, B_0 = 1$and will guarantee that the term $sent$in does not exist $x_k(t), k \geq 1$. Similarly, the solution that reduces to $se_1(t,q)$ when $v = 0$ is obtained by leaving fixed $a_1 = \sqrt{1-\mu^2}$, $A_0 = 1$ and there is no term $cost$ in $x_k(t), k \geq 1$.

In the next problem we will find the second solution by setting $A_0 = 1$ and $a_1 = \sqrt{1-\mu^2}$. The equation for $x_1(t)$ becomes $\frac{d^2x_1}{dt^2} + x_1 = cos3t + B_0 sen3t$, $B_0 = \frac{\mu}{1+\sqrt{1-\mu^2}}$, having as a solution $x_1(t) = -\frac{1}{8}cos3t - \frac{B_0}{8}sen3t + B_1 sent$ for some constant B_1. If we substitute this solution for $x_2(t)$, equating the coefficients of $cost$and $sent$ to zero, we obtain the equations for $a_2 \ _y \ B_1$, whose solutions are $B_1 = 0, a_2 = -\frac{1}{8}$. The solution of this equation is $x_2(t) = B_2 sent - \frac{1}{64}(a_1 + 3\mu B_0)cos3t - \frac{1}{64}(a_1 B_0 - 3\mu)sen3t + \frac{1}{192}(cos5t + B_0 sen5t)$.

If we substitute this solution into the equation for x_3 and by factoring the coefficients of $cost$and $sent$we obtain the equations for B_2 and a_3 giving the solutions $B_2 = \frac{3\mu}{64\sqrt{1-\mu^2}\left(1+\sqrt{1-\mu^2}\right)}, a_3 = -\frac{1+2\mu^2}{64\sqrt{1-\mu^2}}$. Therefore, for $O(q)$you have $x(t) = cost + \frac{\mu sent}{1+\sqrt{1-\mu^2}} - \frac{q}{8}\left(cos3t + \frac{\mu sen3t}{1+\sqrt{1-\mu^2}}\right) + 0(q^2)$.

In the case $\mu = 0$, these series reduce the series for $a_1(q)$ and $ce_1(t,q)$, for the values $q = 1, \mu = v = 0{,}2$, the first series gives $a(q) = 1{,}84$.

Proposition. To $a_0 = 4$ show that if $x_1(t)$ periodic, we have the equation $a_1^2 + a\mu^2 = 0$. Establish that this perturbation method is invalid. Show more generally, that a similar result is obtained if $a_0 = n^2, n \geq 2$.

Indeed. As $a_0 = 4$, by the perturbation method described above we have the following equation $\frac{d^2x_o}{dt^2} + 4x_o = 0$ with the solution $x_0(t) = A_0 cos2t + B_0 sen2t$ and the derivative $\frac{dx_0(t)}{dt} = -2A_0 sen2t + 2B_0 cos2t$. We substitute it into the following equation of the method, the result is obtained:

$$\frac{d^2x_1}{dt^2} + 4x_1 = -(2\mu B_0 + a_1 A_0)cos2t + (2\mu A_0 - a_1 B_0)sen2t -$$

$$A_0 cos4t + B_0 sen4t + A_0 , \qquad (5)$$

and since for (5) $x_1(t)$ is a periodic function, then the system of equations

$$\{a_1 A_0 + 2\mu B_0 = 0 \; 2\mu A_0 - a_1 B_0 = 0 \quad , \qquad (6)$$

we solve (6). The equations are consistent only if the determinant of the system is zero, that is, $det(a_1 \; 2\mu \; 2\mu \; -a_1 \;) = -(a_1^2 + 4\mu^2) = 0$, whose only solution is $\mu = 0 = a_1$, therefore the method does not work. Now in general, for $a_o = n^2$, with $n \geq 2$, the same is done and we have the equation: $\frac{d^2x}{dt^2} + n^2 x_0 = 0$, with the solution $x_0(t) = A_0 cos\,(nt) + B_0 sen(nt)$and the derivative $\frac{dx_0(t)}{dt} = -nA_0 sennt + nB_0 cosnt$ we replace it in the equation of the perturbation method we obtain: $\frac{d^2x_1}{dt^2} + 4x_1 = -(n\mu B_0 + a_1 A_0)cosnt + (n\mu A_0 - a_1 B_o)sennt) - B_0 sen(2t + nt) + A_0 \, cos\, cos\, (-2t + nt) \; + A_0$ $cos\, cos\, (2t + nt) \; + B_0 sen(-2t + nt)$,

and as the problem says, if $x_1(t)$it is a periodic function, then

$$\{a_1 A_0 + n\mu B_0 = 0 \; n\mu A_0 - a_1 B_0 = 0 \quad , \qquad (7)$$

we solve the system (7), and we see that $det(a_1 \; n\mu \; n\mu \; -a_1 \;) = -(a_1^2 + n^2\mu^2) = 0$, whose only solution is $a_1 = 0$ and $\mu = 0$, therefore, it is shown that the method is invalid for $n \geq 2$.

Method rectification: $a_0 = 4$. This last problem shows that an application of perturbation theory fails when $n \geq 2$, can be overcome by fixing $\mu = vq^n$when $a_0 = n^2$. For that matter $n = 2$, we look for a solution that reduces to $ce_2(t,q)$ when $\mu \to 0$. We will start with the general solution for $x_0(t)$, $a_0 = 4$, $x_0(t) = A_0 cos2t + B_0 sen2t$, where A_0, B_0 are constants that must be determined. We look for solutions for $x_k(t), k \geq 1$, that do not contain the term $cos2t$. If we substitute $x_0(t)$ into the equation for $x_1(t)$we find that the equation has a periodic solution only if $a_1 = 0$ and then $x_1(t) = \frac{A_0}{4} - \frac{A_0}{12} cos4t + B_1 sen2t - \frac{B_0}{12} sen4t$.

This expression in the equation for $x_2(t)$ y, setting the coefficients of $cos2t$ y $sen2t$ to zero we obtain the following equations $A_0\left(a_2 - \frac{5}{12}\right) + 2\mu B_0 = 0$, $\left(a_2 - \frac{1}{12}\right) B_0 - 2\mu A_0 = 0$ for a_2 y B_0. If $\mu = 0$ the only non-trivial solutions are $a_2 = \frac{5}{12}$, $A_0 = 1$, $B_0 = 0$ and $a_2 = -\frac{1}{12}$, $A_0 = 0$, $B_0 = 1$, and leads to $ce_2(t,q)$ and $ce_2(t,q)$, respectively. If $\mu > 0$ we can eliminate A_0 and B_0 to give an equation relating a_2 to μ. If we remember that $a = 4 + a_2 q^2$and $v = \mu q^2$, we obtain the following equation for $a(q)$, $4v^2 + \left(a - 4 - \frac{5q^2}{12}\right)\left(a - 4 + \frac{q^2}{12}\right) = 0$. When $v = 0$ this equation is reduced to the curves $a = 4 - \frac{q^2}{12}$, $a = 4 + \frac{5q^2}{12}$, corresponding to $b_2(q)$ and $a_2(q)$, respectively. The equations have two solutions $B_0 = \frac{2\mu A_0}{a_2 + \frac{1}{12}}$, $a_2 =$

$\frac{1}{6}+\frac{1}{4}\sqrt{1-64\mu^2}$, $ce_2(t,q)$ when $\mu=0$, $A_0=\frac{2\mu B_0}{a_2+\frac{5}{12}}$, $a_2=\frac{1}{6}+\frac{1}{4}\sqrt{1-64\mu^2}$, $ce_2(t,q)$ when $\mu=0$, which are real if $0\leq\mu\leq\frac{1}{8}$. From now on we choose the first of these solutions and set $A_0=1$. By perturbation theory, the solution is $x_2(t)=\frac{1}{384}(cos6t+B_0sen6t)+B_2sen2t-\frac{B_1}{12}sen4t$, where B_1 and B_2 are unknown constants. If we substitute $x_2(t)$ into the equation for $x_3(t)$ equating the coefficients of $cos2t$ and $sen2t$ to zero we obtain $B_1=a_3=0$.

A periodic solution for $x_3(t)$ can be found if we substitute into the equation for $x_4(t)$, and we obtain the following linear equations:

$a_4+2\mu B_2=-\frac{\mu B_2}{36}-\frac{19a_2}{144}$, $a_4B_0+2\left(a_2+\frac{1}{12}\right)B_2=\frac{\mu}{36}-\frac{B_0}{4608}-\frac{B_0a_2}{144}$.

These equations can be solved and the process continue. The series obtained for $a(q)$ is $a=4+\frac{q^2}{12}\left(2+3\sqrt{1-64\mu^2}\right)-q^4\left(\frac{763}{13824}+\frac{128}{9}\mu^4+\cdots\right)+0(q^6)$.

Note that the coefficient of q^4 has been expanded into powers of μ to simplify the representation and to show that when $\mu=0, a(q)$.

Mathieu 's functions: are the solutions $\pi$and periodic of 2π Mathieu 's equation $\frac{d^2x}{dt^2}+(a-2qcos2t)x=0$, where a and q are constants which is a Sturm-Liouville system with periodic boundary conditions and has solutions only for particular values of the eigenvalue a, and that depends on q, only periodic solutions are considered. The big difference between Mathieu 's equation and other equations defining special functions is that the coefficient of x is a periodic function of the independent variable, and is the simplest case of the more general equation $\frac{d^2x}{dt^2}+(a+p(t))x=0$, where $p(t)$ is a periodic function of t (Nayteh, 1995).

Mathieu functions is complex, particularly when we have to understand that $t$and q depend on all eigenfunctions. More details of the study with Maple can be found in Kreyszig (1994); Richards (2002). The variable q is a parameter that we need to get the behavior of both eigenvalues and both eigenfunctions as a function q. In this special case there are $q=0$ periodic solutions only if $a=n^2, n=0,1,2,3,\ldots$ and these solutions are $1, cost,\ cos2t,\ldots$.(even solutions), $sent,\ \ sen2t,\ldots$(odd solutions).

Mathieu functions corresponding to values of q, when $q\to\infty$ denoted as follows: $ce_0(t,q),\ ce_1(t,q)$, $ce_2(t,q)$, ... (even solutions), $ce_1(t,q),\ ce_2(t,q)$, ..., (odd solutions).

If $q\neq0$ each of these eigenfunctions has a different eigenvalue. For each eigenvalue there exists at most one period solution π o 2π, and each pair $\{ce_n(t,q), ce_n(t,q)\}$ has n zeros in the interval $-\pi<t\leq\pi$. These solutions can be unique in several ways, one of these ways is by choosing the coefficient of $cosnt$ en $ce_n(t,q)$ and the coefficient of $sent$ en $ce_n(t,q)$. The eigenvalue associated with the even solutions, $ce_k(t,q)$, is denoted by $a_k(q), k=0,1,2,3,\ldots$, and the eigenvalue associated with the odd solutions, $ce_k(t,q)$, as $b_k(q), k=0,1,2,3,\ldots$, Table 1.

Table 1

Mathieu functions

Function	Period	Parity environment $t=0$	Parity environment$t=\frac{\pi}{2}$	Eigenvalues
$ce_{2r}(t,q)$ $ce_{2r+1}(t,q)$	π 2π	Pair	Pair Odd	$a_{2r}(q)$,$r=0,1,\ldots$ $a_{2r+1}(q)$,$r=0,1,\ldots$

$ce_{2r}(t,q)$ $ce_{2r-1}(t,q)$	π 2π	Odd	Odd Pair	$b_{2r}(q), r = 0,1,\ldots$ $b_{2r-1}(q), r = 0,1,\ldots$

Source: self made.

Mathieu functions

Mathieu 's equation is a Sturm-Liouville system $\frac{d}{dt}\left(p(t)\frac{dy}{dt}\right) + (q(t) + \lambda w(t))y = 0$, with $p(z) = 1, q(z) = -2qcos2t, w(t) = 1, \lambda = a$ and with periodic boundary conditions.

Considering the space L^2, the set of eigenfunctions $\{ce_r(t,q)\}$is $\{se_r(t,q)\}$ complete on the interval $-\pi \leq t \leq \pi$. Additionally, each of the sets $\{se_r(t,q)\}$ y $\{se_r(t,q)\}$ is complete in $0 \leq t \leq \pi$ and each of the sets $\{se_{2r}(t,q)\}$ y $\{se_{2r+1}(t,q)\}$, $\{se_{2r}(t,q)\}$ y $\{se_{2r+1}(t,q)\}$ is complete in the interval $0 \leq t \leq \pi/2$.

Mathieu functions are orthogonal $\int_{-\pi}^{\pi} ce_m(t,q)ce_n(t,q)dt = h_n\delta_{nm}$,

$\int_{-\pi}^{\pi} ce_m(t,q)ce_n(t,q)dt = 0$, the normalization constant $h_k = \pi$.

The eigenvalues are ordered $a_0(q) < b_1(q) < a_1(q) < b_2(q) < a_2(q) < \cdots \quad (q \neq 0)$.

It is convenient to divide eigenvalues into three types: eigenvalues with values much smaller than $2q$, eigenvalues whose values are much larger than , $2q$ and eigenvalues that are around $2q$.

For large eigenvalues , for which $a - 2qcos2t$ is always positive, we have $a_r(q) \to b_r(q) \to r^2$ when $q \to 0$. For q fixed and r large, the difference between a_r and b_r, is exponentially small, $a_r(q) - b_r(q) = 0\left(\frac{q^r}{r^{r-1}}\right)$ when $r \to \infty$.

Large $r$eigenvalues can be approximated by the following $a_r(q) \cong b_r(q) \sim r^2 + \frac{q^2}{2r^2} + \cdots$series . Yes $q \gg 1, a \ll q$, therefore $a - 2qcos2t$ it changes sign, $b_{r+1}(q)$ and $a_r(q)$ they are close.

For $|q|$ small values, the eingenvalues can be expanded as a power series in q, which can be found using perturbation theory, but a clear view of the relationship between the $a_r(q)$ and the $b_r(q)$, is seen in the following figure, in which the first twelve values eigenvalues are shown for $0 < q < 35$. The order of the curves in this figure can be understood if we note that $a_0 = \frac{q^2}{2} + O(q^4)$, for $r \geq 1$, $a_r(0) = b_r(0) = r^2$ and for $q > 1, a_r(q) > b_r(q)$, and for $q \gg 1, a_r(q) \cong b_{r+1}(q)$.

If $q < 0$the change of variables $t \to \frac{\pi}{2} - t$ changes the sign of in $cos2t$ Mathieu 's equation, then the following relationship holds $a_{2r}(-q) = a_{2r}(q)$: , $b_{2r}(-q) = b_{2r}(q)$, $a_{2r+1}(-q) = b_{2r+1}(q)$, and

$ce_{2r}(t,-q) = (-1)^r ce_{2r}\left(\frac{\pi}{2} - t, q\right)$,

$ce_{2r+1}(t,-q) = (-1)^r ce_{2r+1}\left(\frac{\pi}{2} - t, q\right)$,

$ce_{2r+1}(t,-q) = (-1)^r ce_{2r+1}\left(\frac{\pi}{2} - t, q\right)$,

$ce_{2r}(t,-q) = (-1)^{r-1} ce_{2r}\left(\frac{\pi}{2} - t, q\right)$.

Eigenfunctions with eigenvalues $a > 2q$ They oscillate uniformly. However, there is a change in the relationship between the even and odd functions, if $a \gg 2q, b_r(q) \cong a_r(q)$, while if $a \ll 2q, a_{r+1}(q) \cong b_r(q)$.

CONCLUSION

The analysis of Mathieu's equation forces us to go through many preponderant topics for its study. First, understand how to arrive at Mathieu's equation and what type of equation it is. Second, discover that the Mathieu equation cannot be solved by elementary functions, have the motivation to solve the equation with new special functions that we know as Mathieu functions, which have an extensive theory. For the analysis of the behavior of the Mathieu equation and its approximate analytical solutions, and it was necessary to solve using perturbation theory, since this is a method of analytical approximations and was done for the Mathieu equation but with a small linear damper term, We arrived at solutions, compared them with exact solutions and noticed that the method we used was very good due to the great similarity of both.

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